PRECISE POINT POSITIONING FOR HIGH PRECISION GEODETIC APPLICATIONS

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1st International Conference:
NEW ADVANCED GNSS AND 3D SPATIAL TECHNIQUES APPLICATIONS to CIVIL and ENVIRONMENTAL ENGINEERING, GEOPHYSICS, ARCHITECTURE, ARCHEOLOGY and CULTURAL HERITAGE

Trieste, 18 – 20 February 2016

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Introduction

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  - Proper set of unknowns (*coordinates, troposphere parameters, receiver clock errors, phase ambiguities, differential code biases*)
  - Elimination/mitigation of GNSS biases to a mm level + cycle slips detection and removal
  - Highly precise/accurate IGS products

Development of PPP

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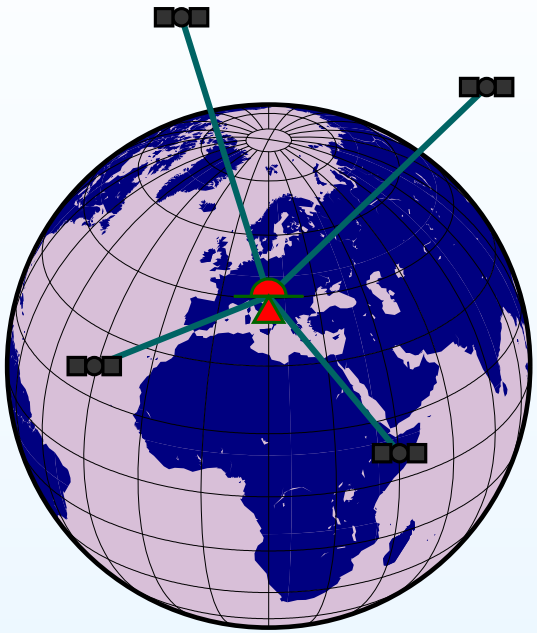
- Ambiguity fixed PPP
  - Network solution only or
  - Additional external information

Principle of PPP



Receiver tracks signal from k satellites (each epoch)

$$
\begin{aligned}
L_1 &= \rho(x,y,z) + \tau_R + T - I + N_1 + \xi_L + \varepsilon_{L_1} \\
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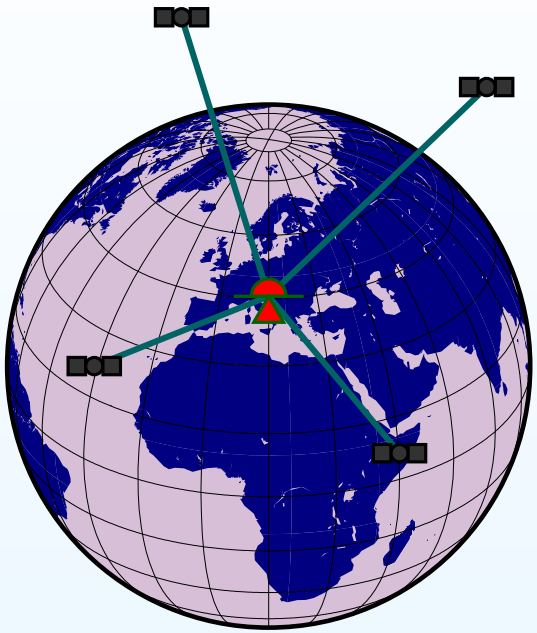

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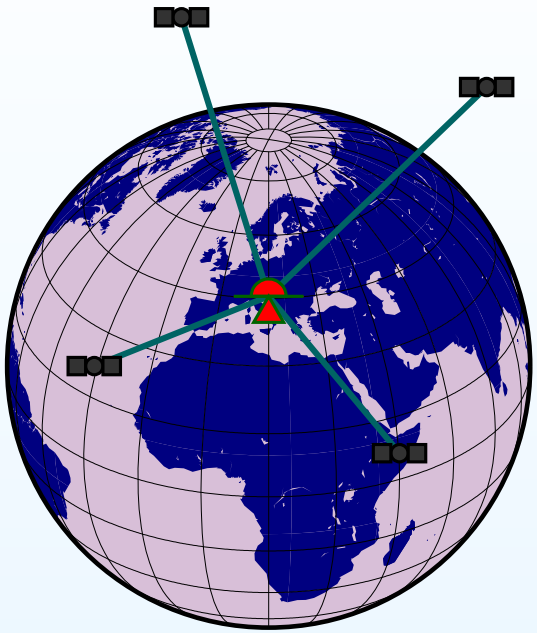
- Observations: L_1, L_2, P_1 and P_2 ([m])

Principle of PPP



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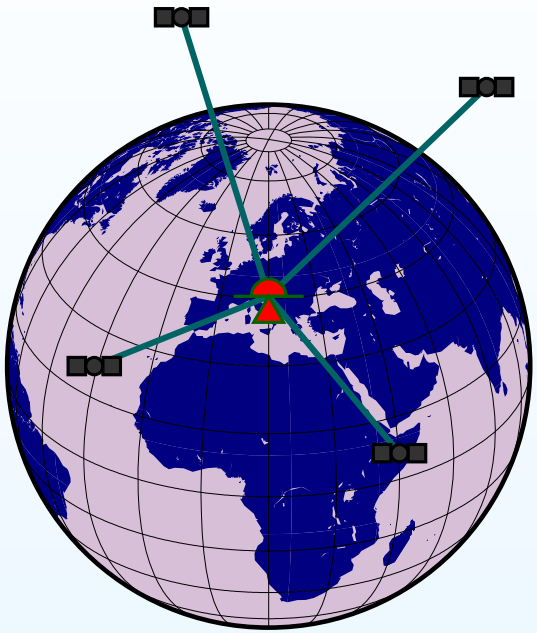

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- Observations: L_1, L_2, P_1 and P_2 ([m])
- Unknowns: $\rho(x,y,z)$, τ_R, N_i, D_i,
- Biases:
  - models: ξ_P and ξ_L
  - $\sim$approksimation: I and T

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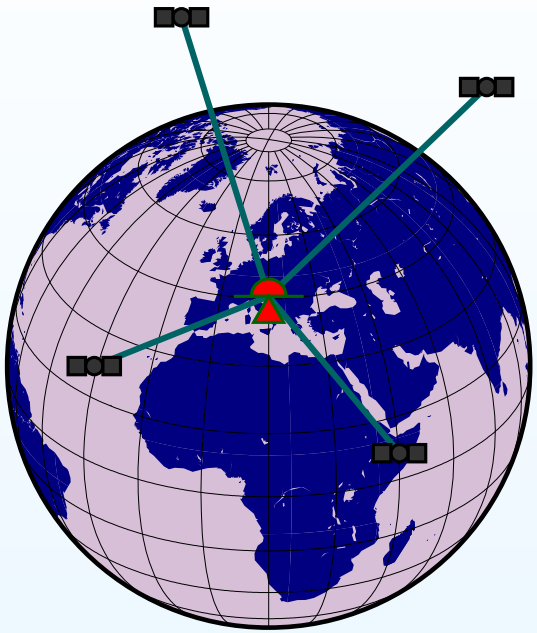

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- Random errors and unmodeled biases: ε_i

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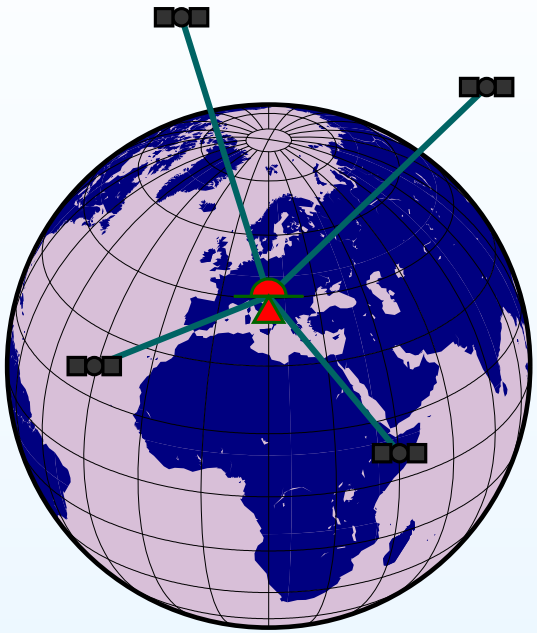
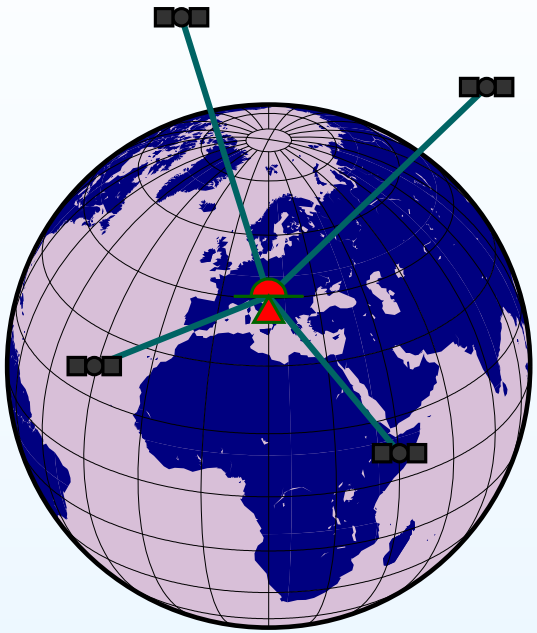


Preparing observation for processing:

Principle of PPP



Preparing observation for processing:

1. Modelling ξ_P and ξ_L

2. $T = T_d + \overbrace{M_w T_w^z + M_g \left(G_N \cos\alpha + G_E \sin\alpha\right)}^{T_m}$
   introduction of unknowns: T_w^z, G_N in G_E

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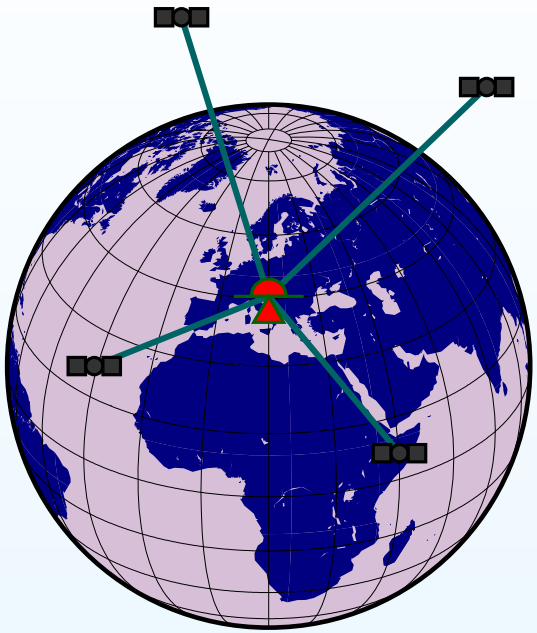

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$$L_3 = \rho + \tau_R + T_m + N_3 + \varepsilon_{L_3}$$

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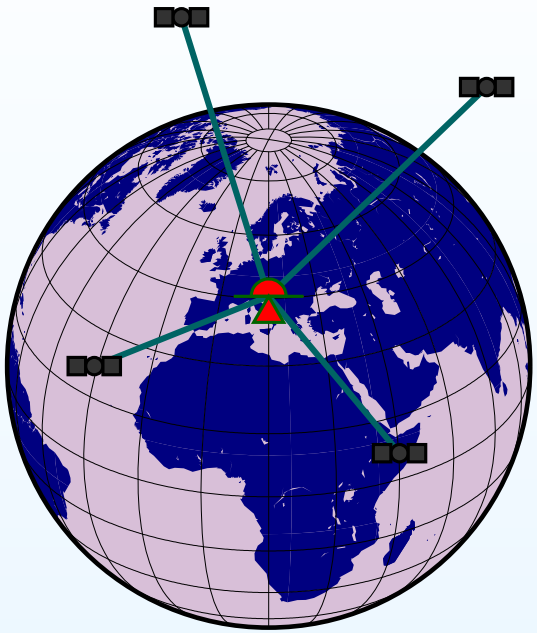

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4. Setting up phase ambiguities (cycle slip determination)

Geometry of PPP

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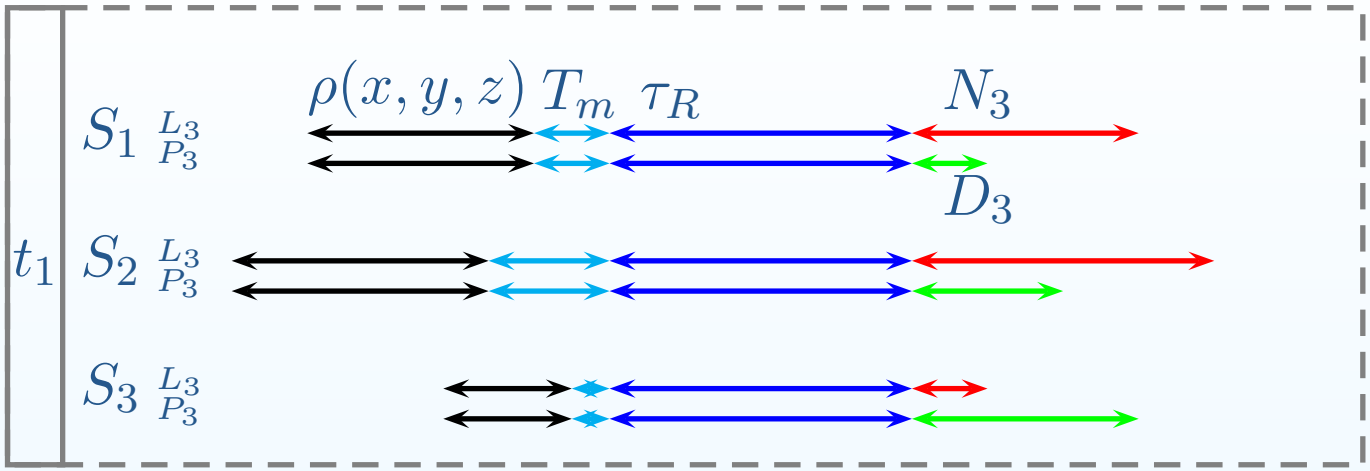

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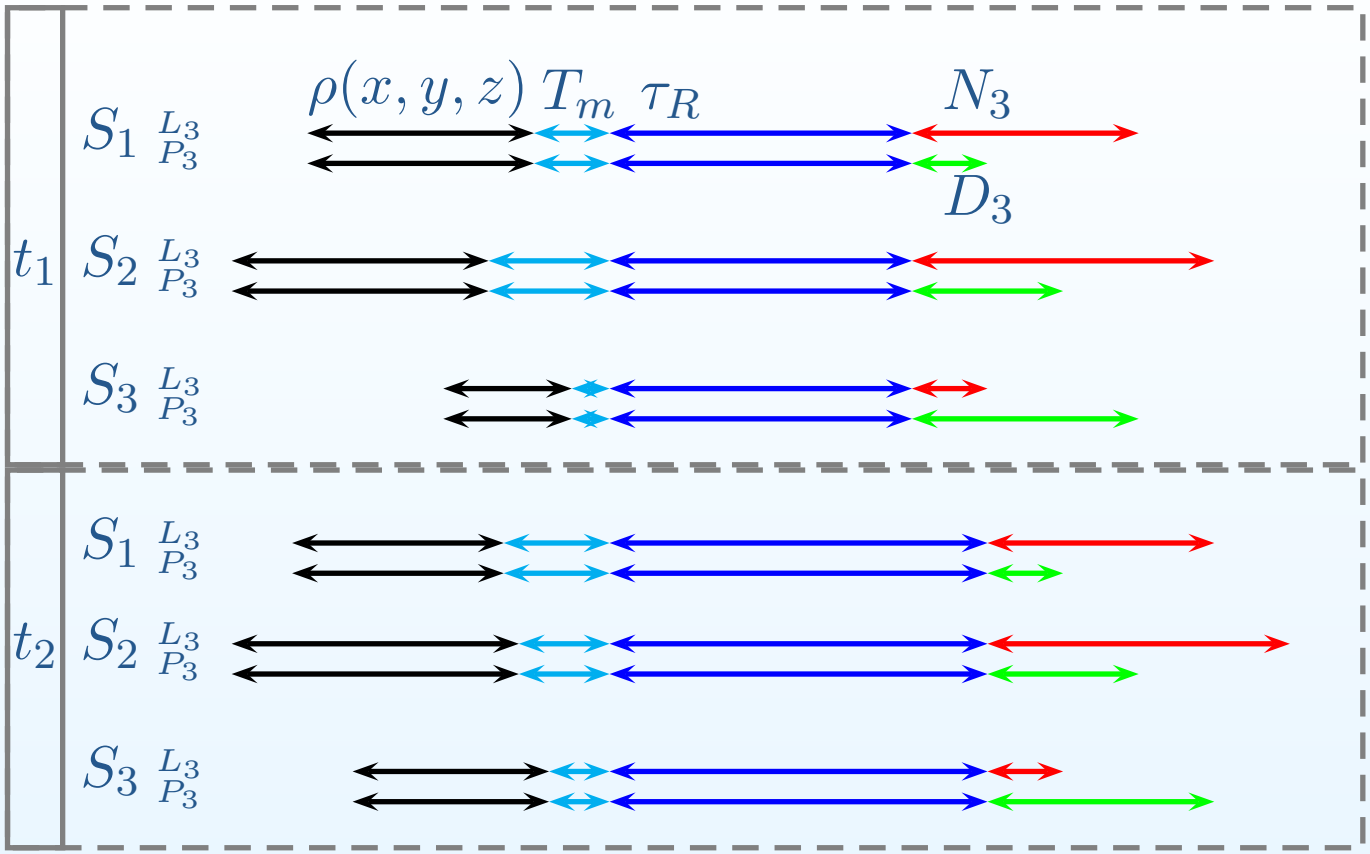

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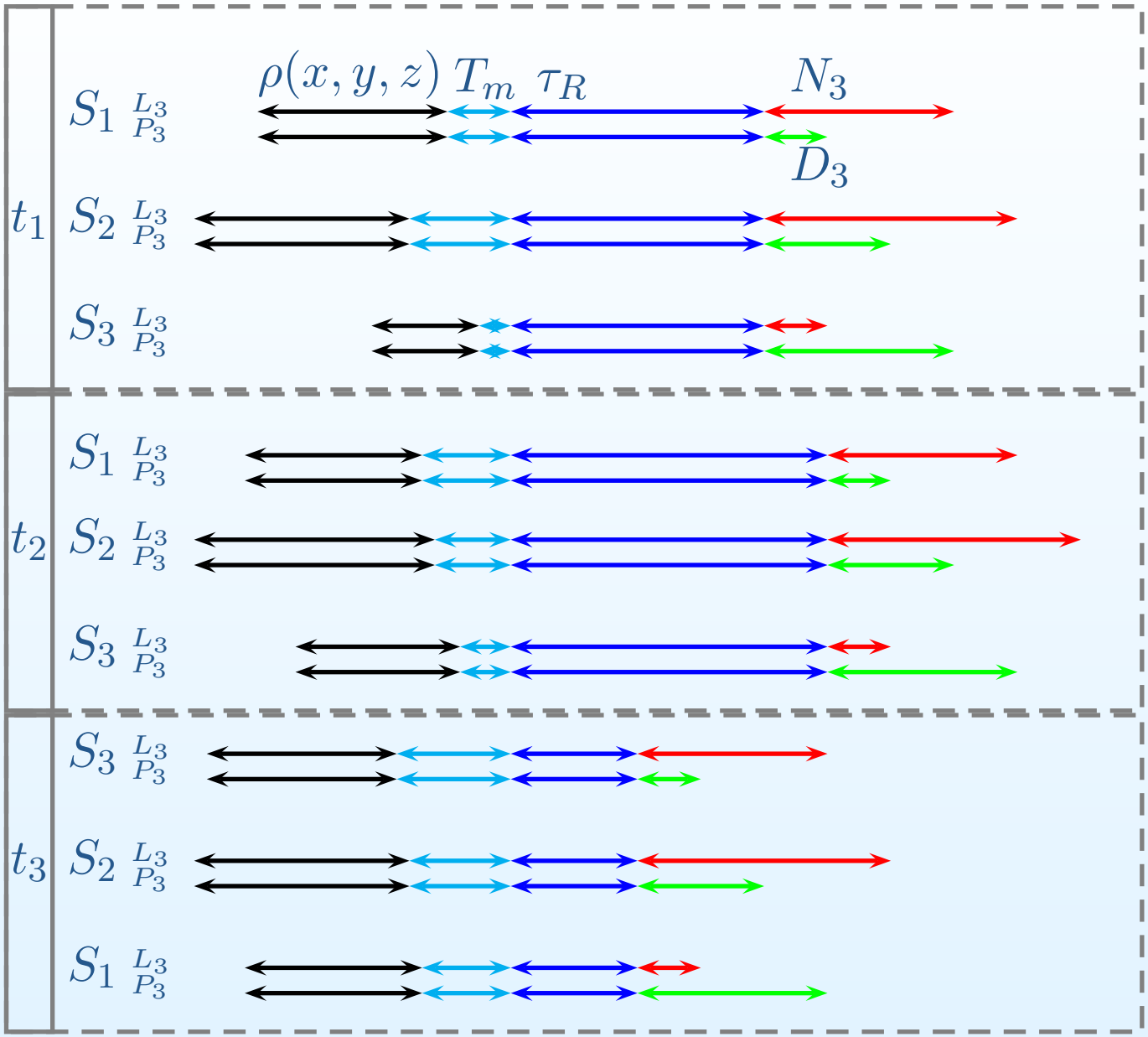

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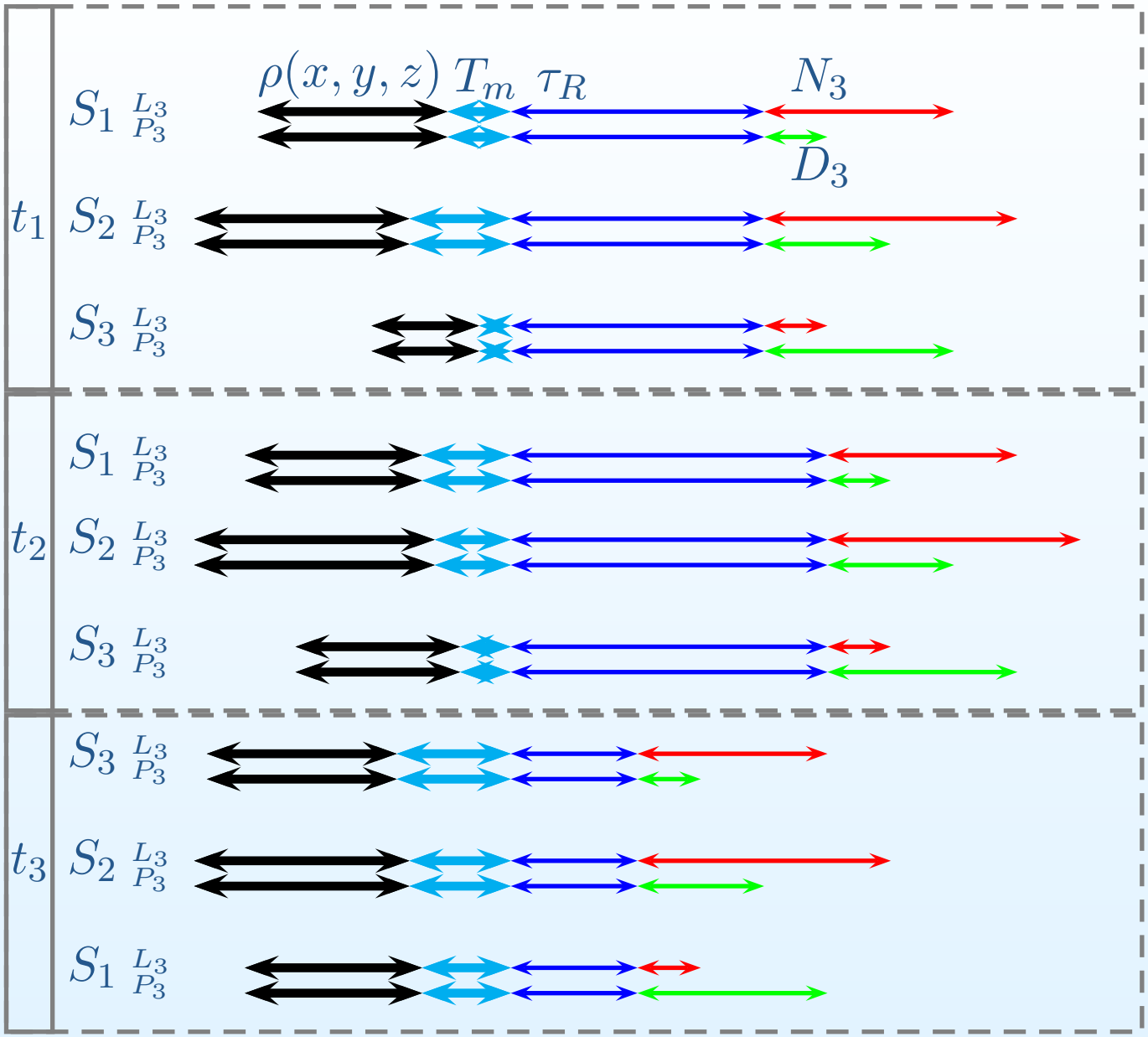

$\rho(x,y,z)$ and T_m:

– nonlinear

– variable (S_i, t_i)

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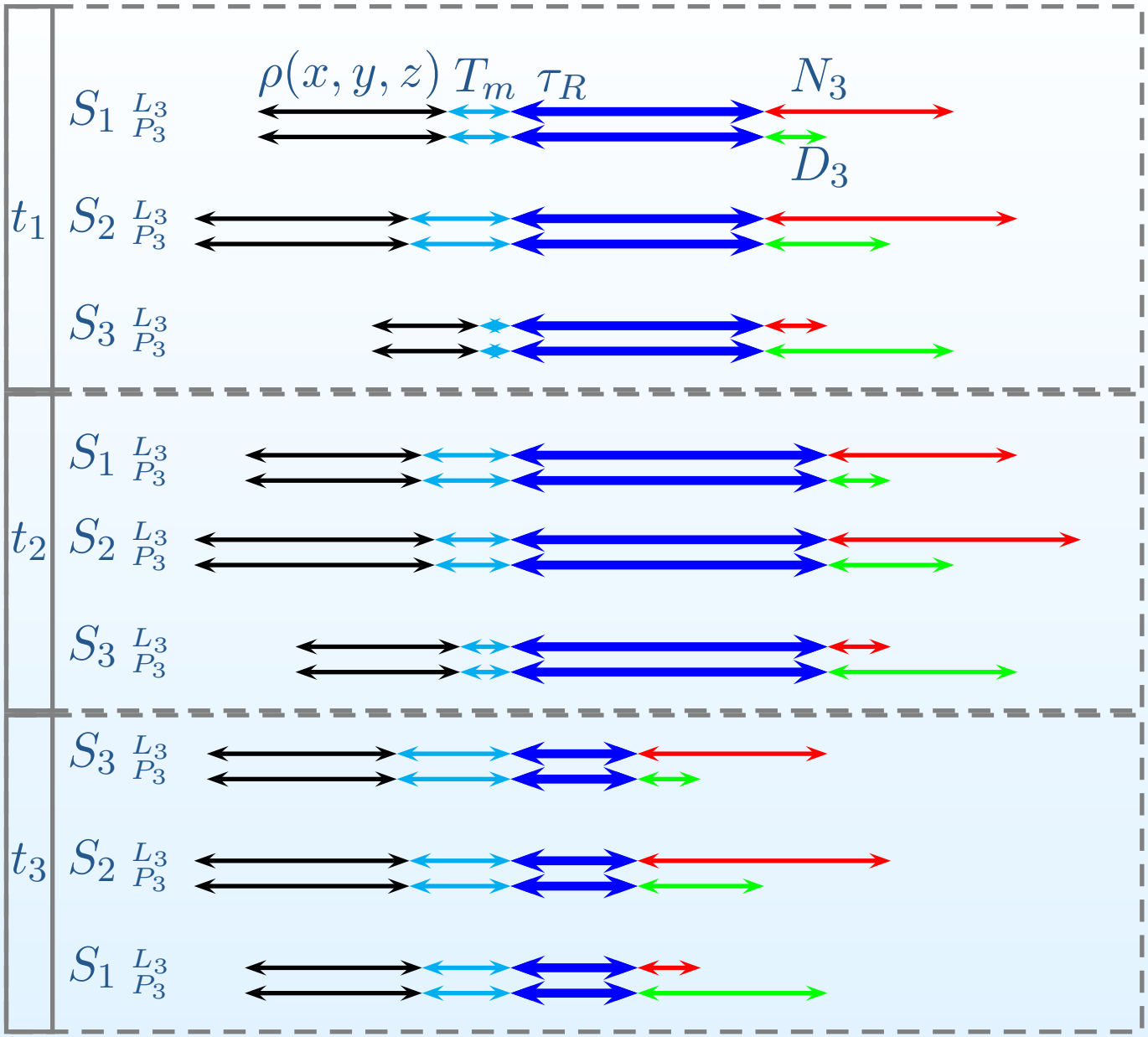

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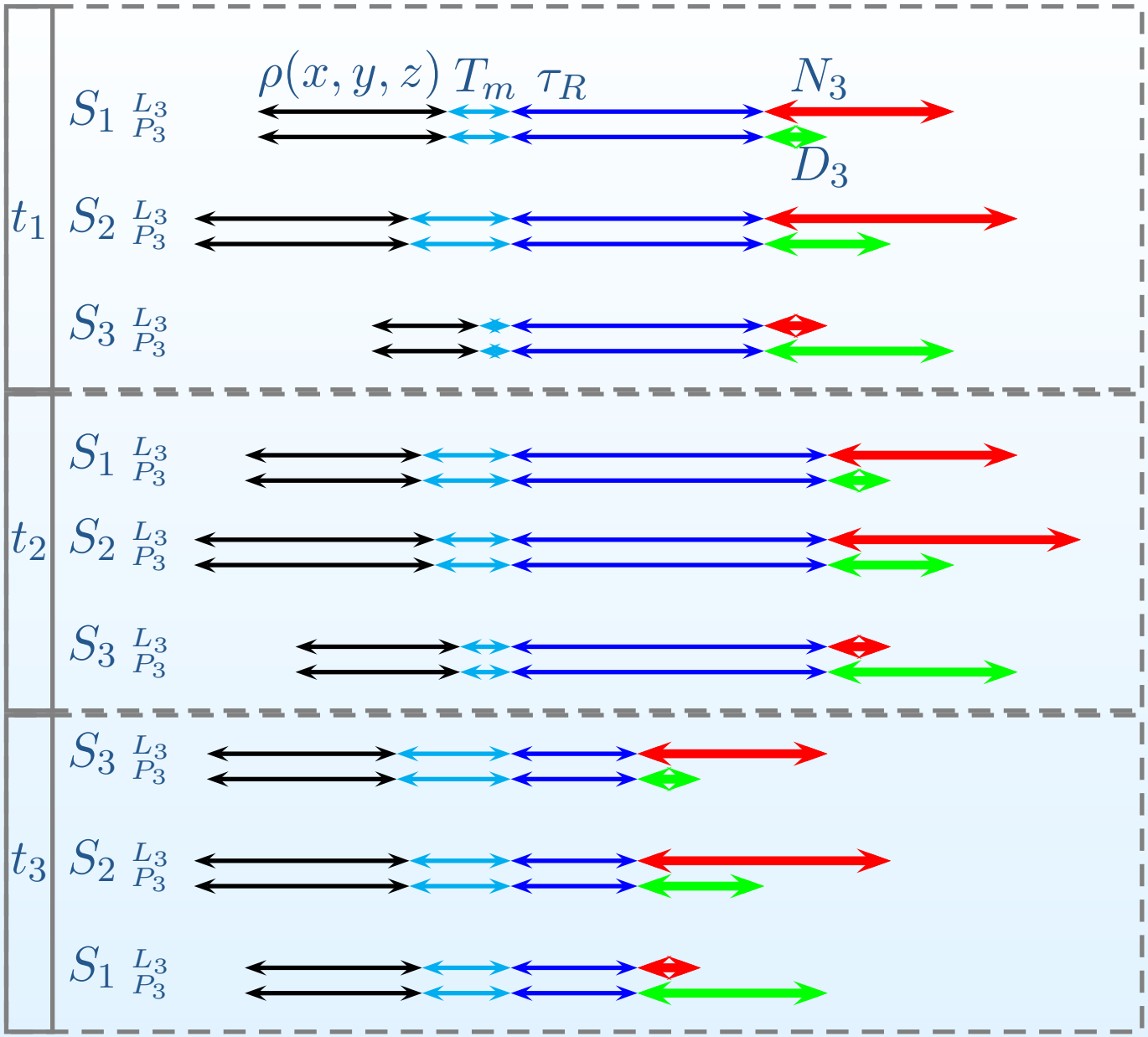

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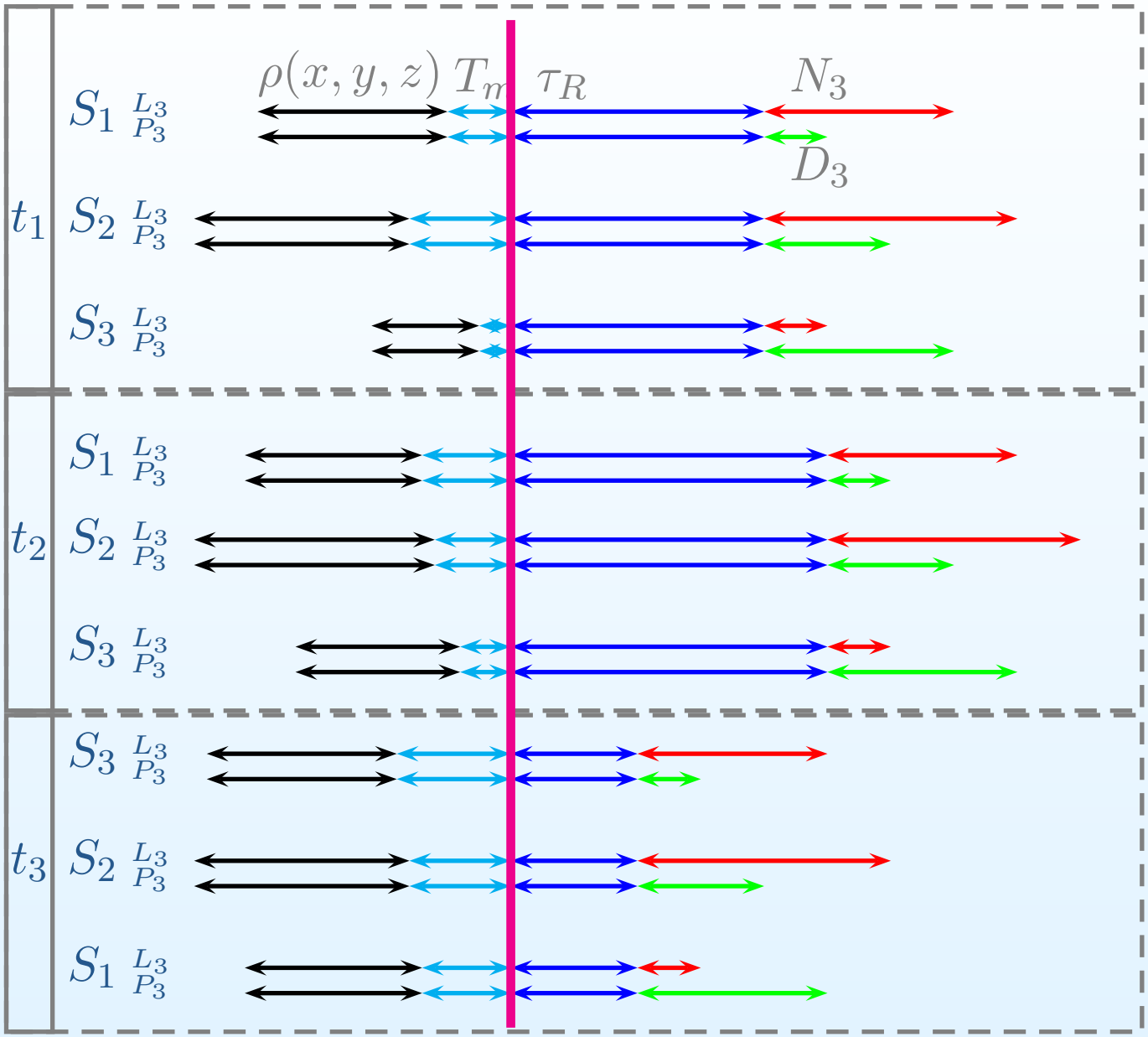


2 types of unknows:

– nonlinear (left)

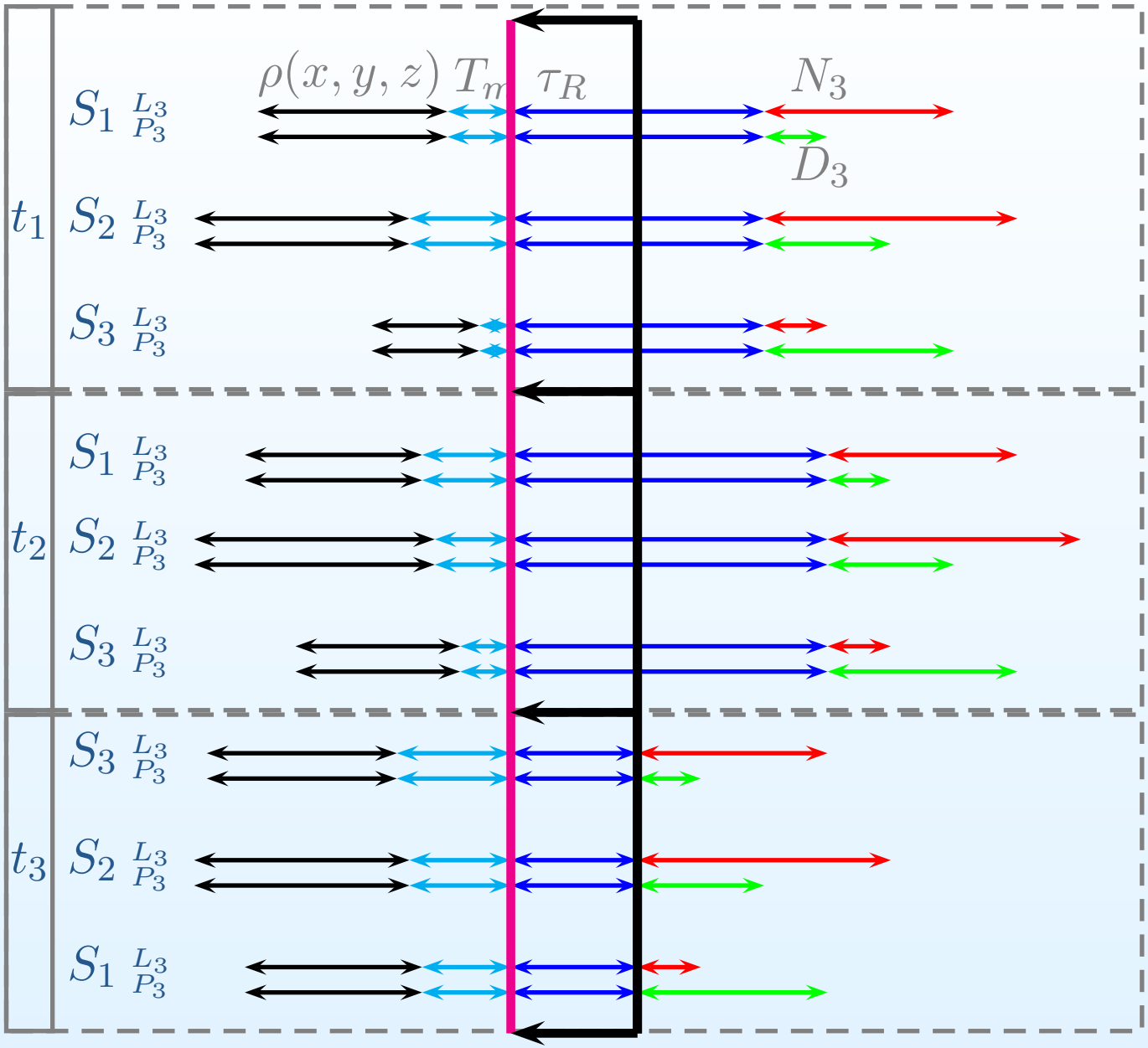
– linear (right)

Left unknowns:

– biased trilateration

– observable

Geometry of PPP



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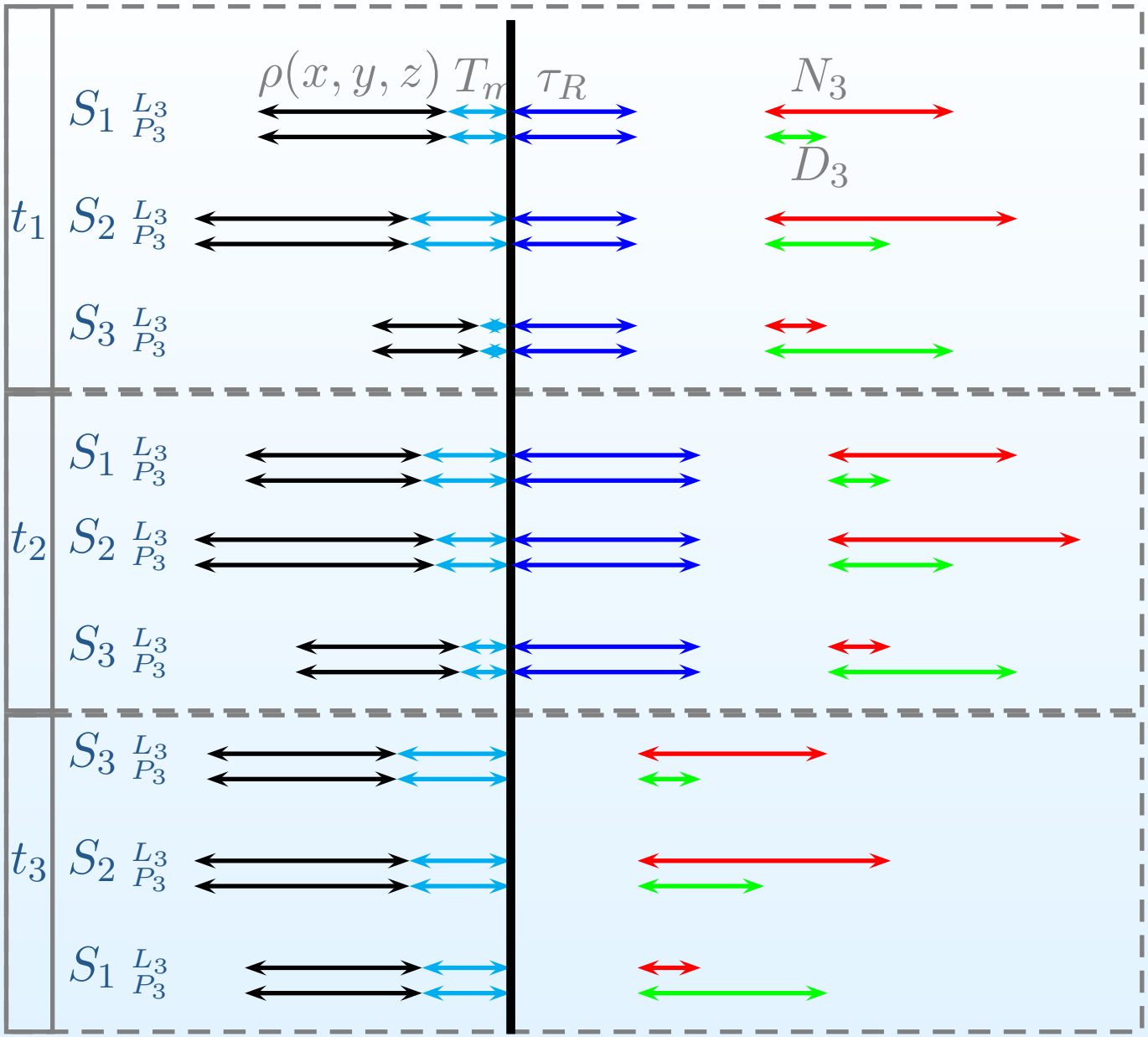

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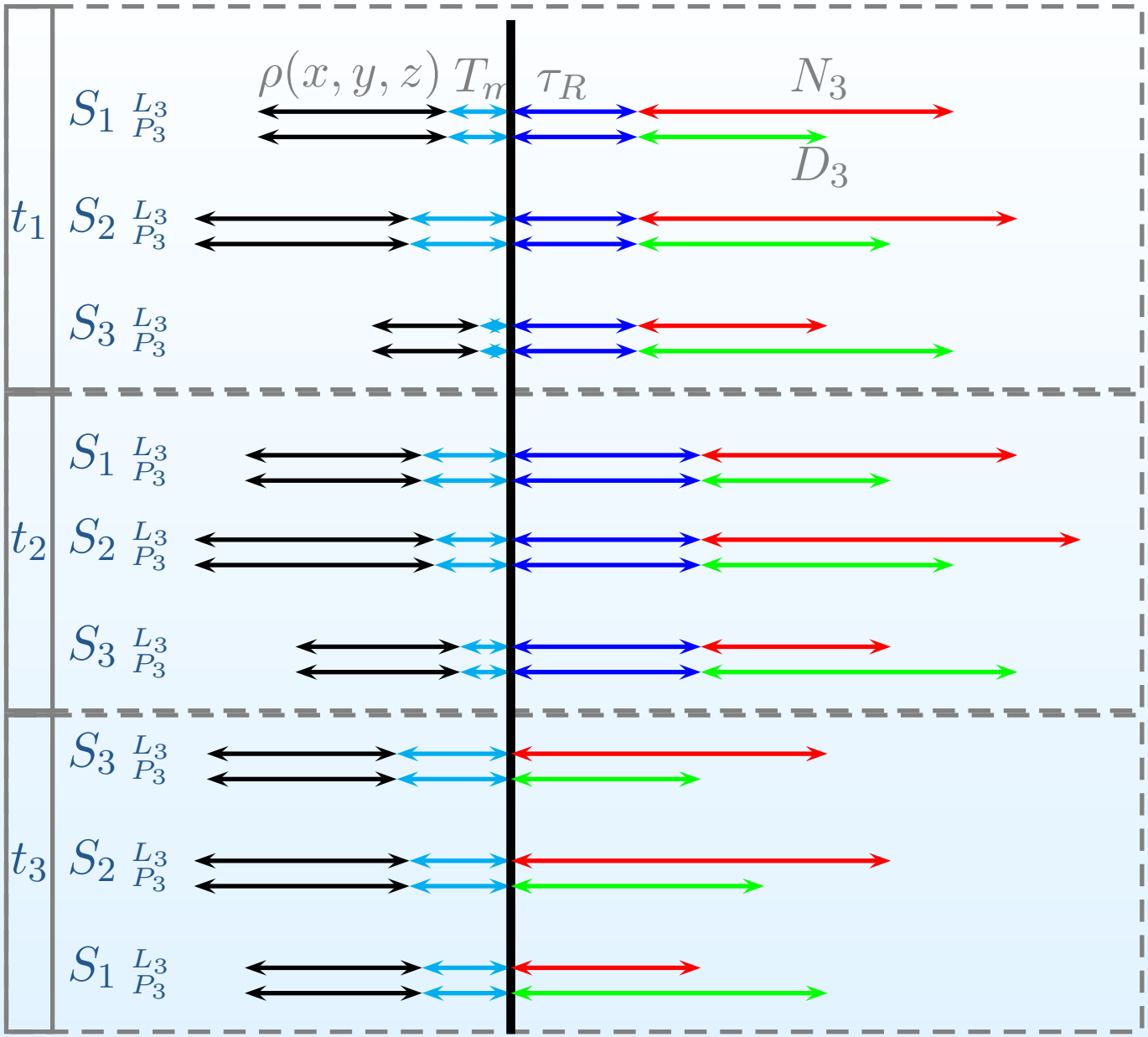

2 types of unknows:

– nonlinear (left)

– linear (right)

Left unknowns:

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– observable

Right unknowns:

– changeable

– observable? **NO**

Gauss-Markov model of PPP

Functional model – linearized observation equations

$$
\begin{aligned}
\mathrm{E}(L_3 - L_{3,0}) &= \frac{\partial L_3}{\partial \rho(x,y,z)}\delta\rho + \frac{\partial L_3}{\partial T_m}\delta T_m + \frac{\partial L_3}{\partial \tau_R}\delta\tau_R + \frac{\partial L_3}{\partial N_3}\delta N_3 \\
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All observed L_3 and P_3 from all satellites and all epochs into:

$$\mathrm{E}(\mathbf{f}) = \mathbf{B}\boldsymbol{\Delta}$$

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Stochastic model – $\sigma_{P_3} = 100\sigma_{L_3}, p(e) = \cos(e)$

Gauss-Markov model

Gauss-Markov (GM) linearized model

$$E\underbrace{\begin{bmatrix} \mathbf{f}_{L,1} \\ \mathbf{f}_{P,1} \\ \vdots \\ \mathbf{f}_{L,n} \\ \mathbf{f}_{P,n} \end{bmatrix}}_{\mathbf{f}} = \underbrace{\begin{bmatrix} \mathbf{B}_1^{\rho} & \mathbf{B}_1^{T_m} & \mathbf{B}_1^{N} & \mathbf{0} & \mathbf{1} & \cdots & \mathbf{0} \\ \mathbf{B}_1^{\rho} & \mathbf{B}_1^{T_m} & \mathbf{0} & \mathbf{B}_1^{D} & \mathbf{1} & \cdots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ \mathbf{B}_n^{\rho} & \mathbf{B}_n^{T_m} & \mathbf{B}_n^{N} & \mathbf{0} & \mathbf{0} & \cdots & \mathbf{1} \\ \mathbf{B}_n^{\rho} & \mathbf{B}_n^{T_m} & \mathbf{0} & \mathbf{B}_n^{D} & \mathbf{0} & \cdots & \mathbf{1} \end{bmatrix}}_{\mathbf{B}} \underbrace{\begin{bmatrix} \boldsymbol{\Delta}_{\rho} \\ \boldsymbol{\Delta}_{T_m} \\ \boldsymbol{\Delta}_{N} \\ \boldsymbol{\Delta}_{D} \\ \delta\tau_{R,1} \\ \vdots \\ \tau_{R,n} \end{bmatrix}}_{\boldsymbol{\Delta}}$$

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- Solution to GM model:
  - Consistency:
    $\mathrm{E}(\mathbf{f}) = \mathbf{B}\boldsymbol{\Delta} \quad \rightarrow \quad \mathbf{f} = \mathbf{B}\boldsymbol{\Delta} + \mathbf{v} \quad \rightarrow \quad \mathbf{v}^{\mathrm{T}}\mathbf{P}\mathbf{v} \rightarrow min.$
  - Uniqueness: $r(\mathbf{B}) = u$ (u-number of unknowns)

Uniqueness of Gauss-Markov model

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- Geometrical representation of $\mathsf{N}(\mathbf{B})$:
$$\sum \delta N + \sum \delta D - \sum \delta\tau_R = 0$$

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- Particular solution with $\mathbf{w}^{\mathrm{T}}\boldsymbol{\Delta} = 0 \rightarrow$ S-transformation

$$\mathbf{S} = \mathbf{I} - \mathbf{y}\left(\mathbf{w}^{\mathrm{T}}\mathbf{y}\right)^{-1}\mathbf{w}^{\mathrm{T}} \quad \rightarrow \quad \begin{aligned}\boldsymbol{\Delta}_w &= \mathbf{S}\boldsymbol{\Delta}_h \\ \mathbf{Q}_{\boldsymbol{\Delta}w} &= \mathbf{S}\mathbf{Q}_{\boldsymbol{\Delta}h}\mathbf{S}^{\mathrm{T}}\end{aligned}$$

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- Unbiased unknowns: x, y, z, T_m
- Biased unknowns: N, D, τ_R

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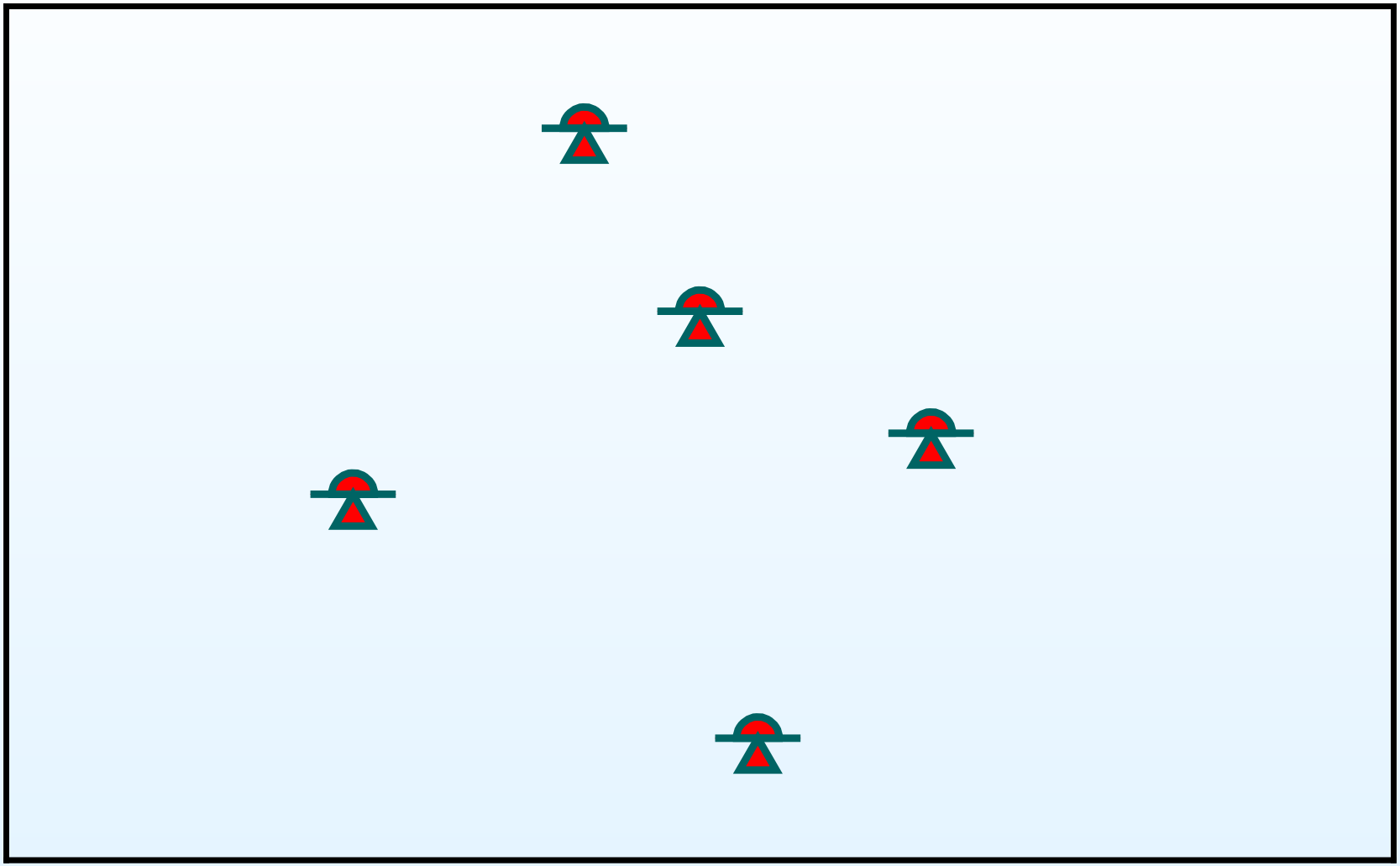
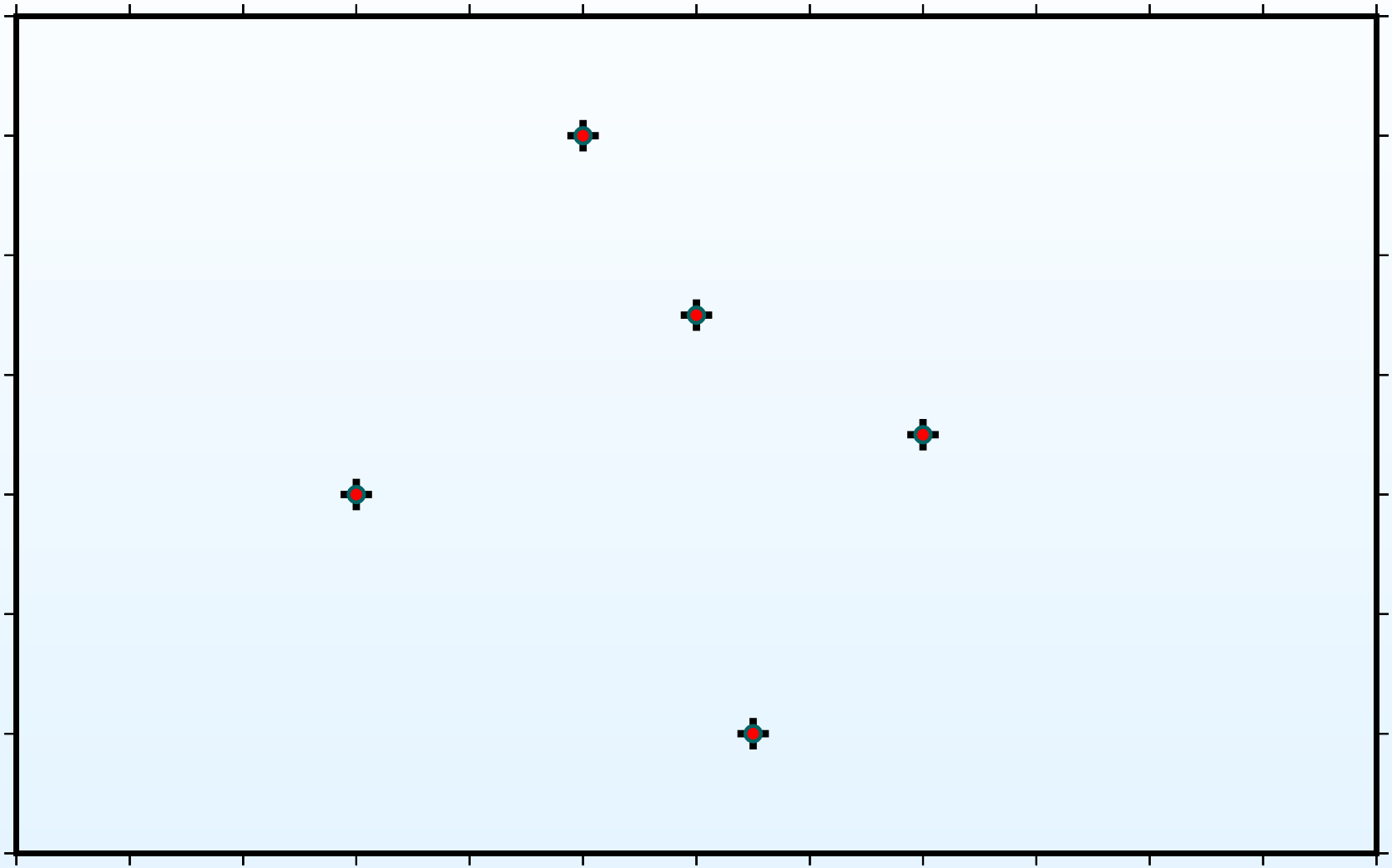
- Consequence:
  - Calculation of any $\mathbf{\Delta}_w$ possible
  - Calculation of $\mathbf{Q}_{\mathbf{\Delta}_w}$ NOT possible

Uniformity of PPP cordinates



PPP survey on a set of new geodetic points

Uniformity of PPP cordinates



Estimated PPP coordinates in a global coordinate system

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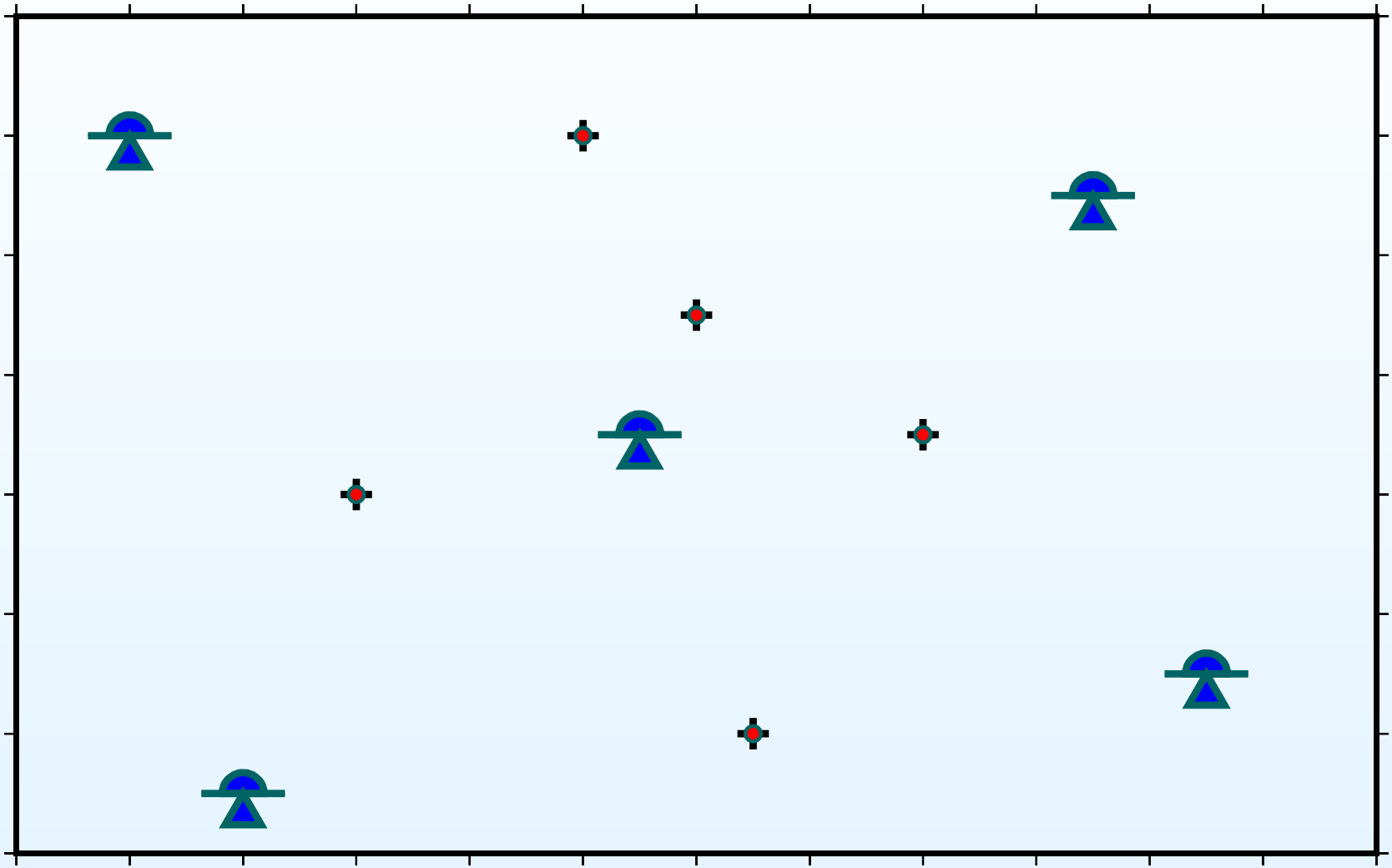

Quality control - reference stations are included in survey (known coordinates)

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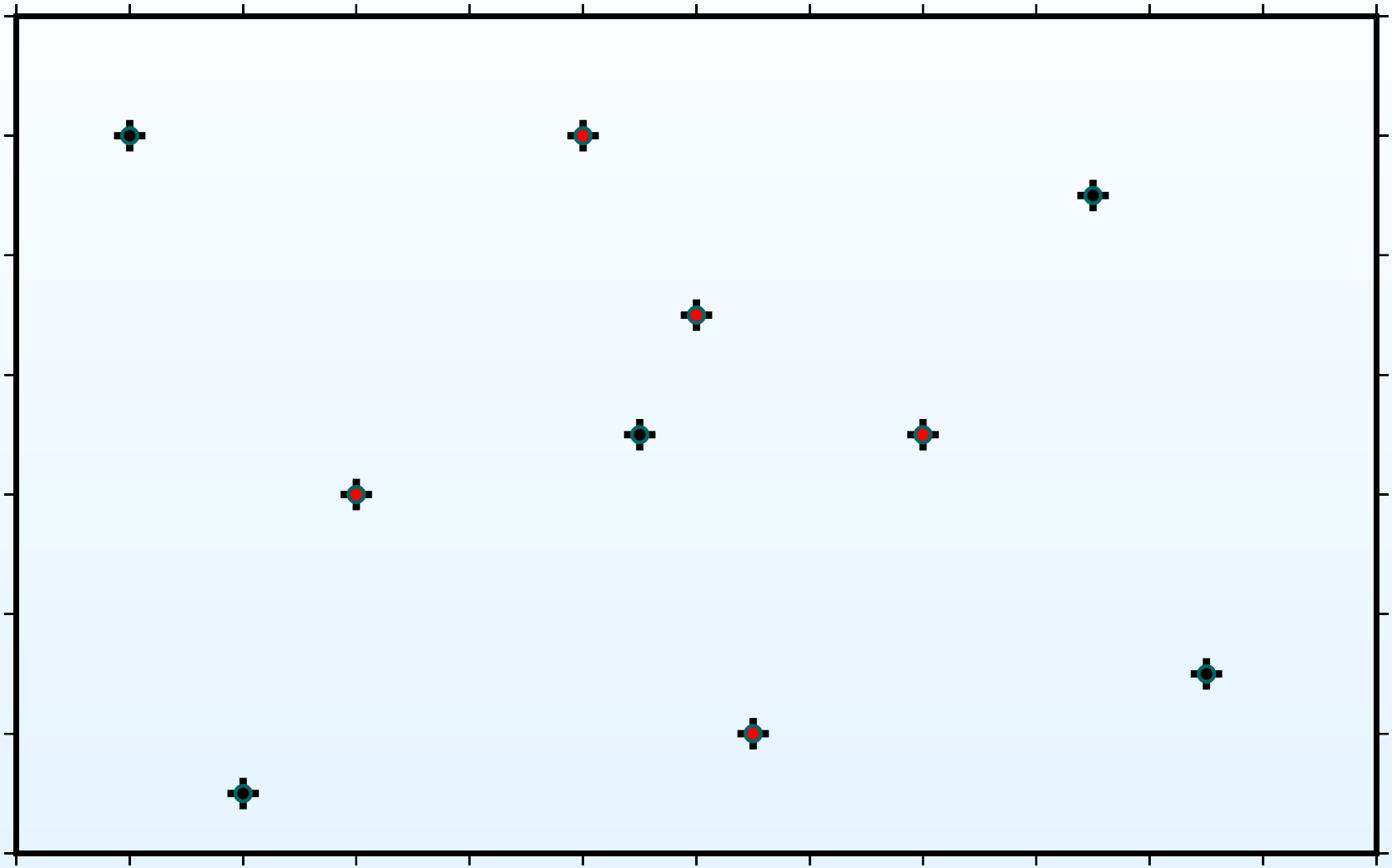

Estimated PPP coordinates of all geodetic points in global coordinate system

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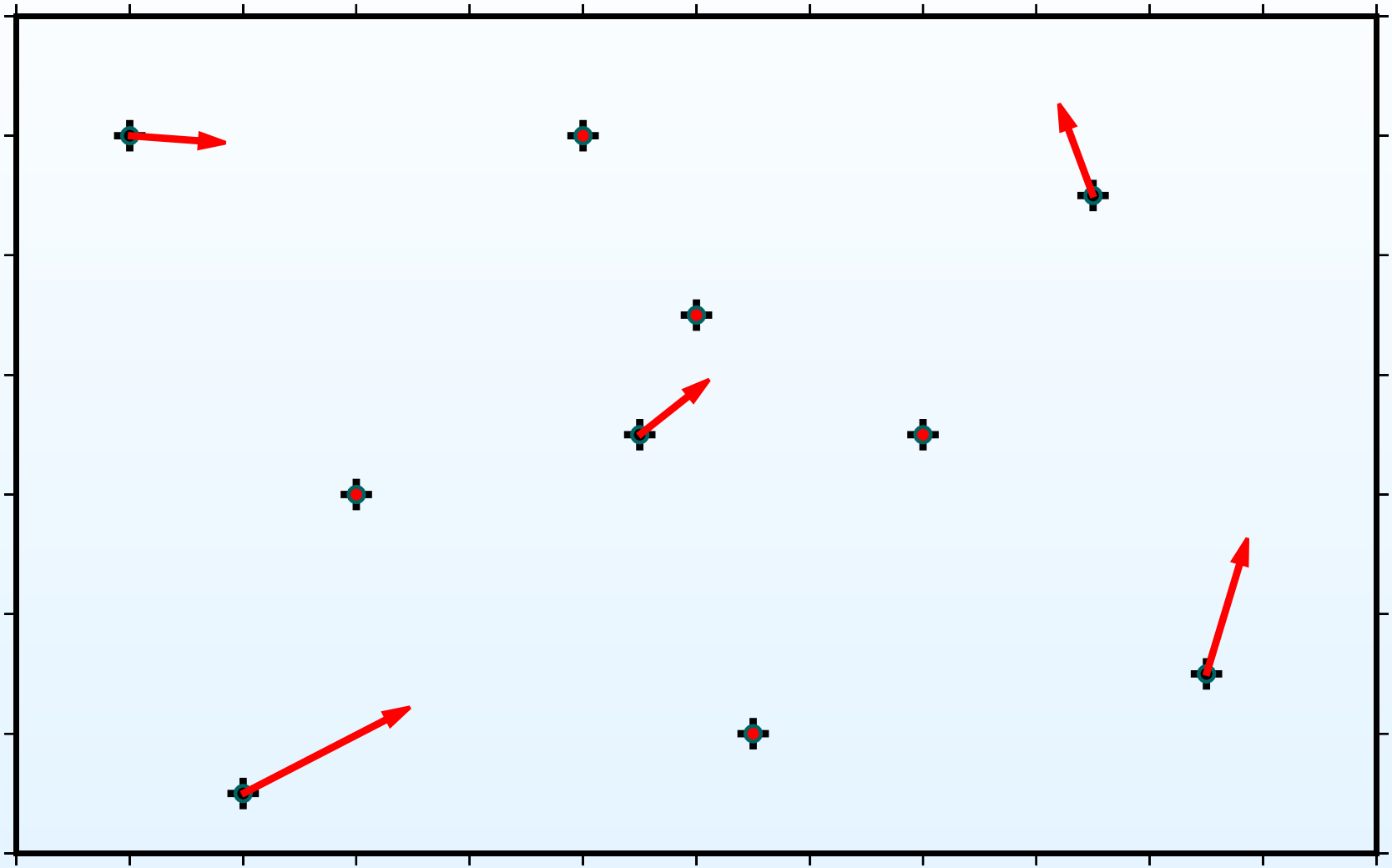

Differences of PPP coordinates with respect to reference coordinates

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- Removal of systematic component:
  - spatial similarity transformation
  - presumption: small values of transformation parameters

. . . representation of PPP coordinates in uniform global coordinate system.

Similarity transformation

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- Functional model:

$$\mathbf{X}^P = \mathbf{X}^I + \mathbf{T} + m\mathbf{X}^I + \mathbf{R}(\omega_x, \omega_y, \omega_z)\mathbf{X}^I$$

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- Final mathematical model:

$$\mathbf{v} + \mathbf{M}\beta = \mathbf{f} \qquad \mathbf{P} = \begin{bmatrix} \mathbf{P}_1 & \cdots & \mathbf{0} \\ \vdots & \ddots & \vdots \\ \mathbf{0} & \cdots & \mathbf{P}_{n_p} \end{bmatrix}$$

- Matrix $\mathbf{M}$: the inner constraints matrix

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$$\beta = \left(\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{M}\right)^{-1}\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{f} \qquad \mathbf{Q}_{\beta} = \left(\mathbf{M}^{\mathrm{T}}\mathbf{P}\mathbf{M}\right)^{-1}$$

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- We get: $\mathbf{v} = \mathbf{S}\mathbf{f} \quad \mathbf{Q}_{\mathbf{v}} = \mathbf{S}\mathbf{Q}^P\mathbf{S}^{\mathrm{T}}$ **S-TRANSFORMATION**

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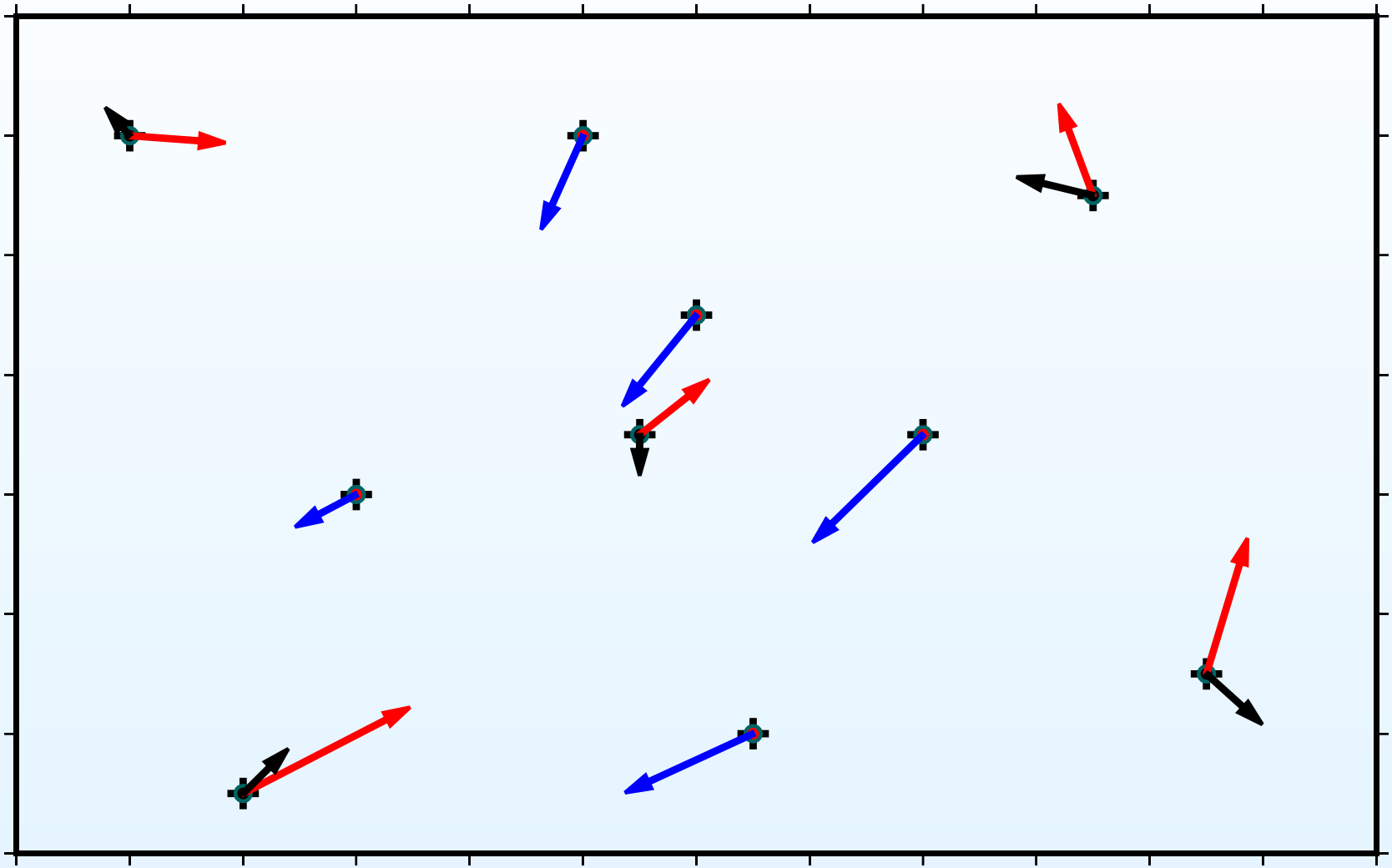
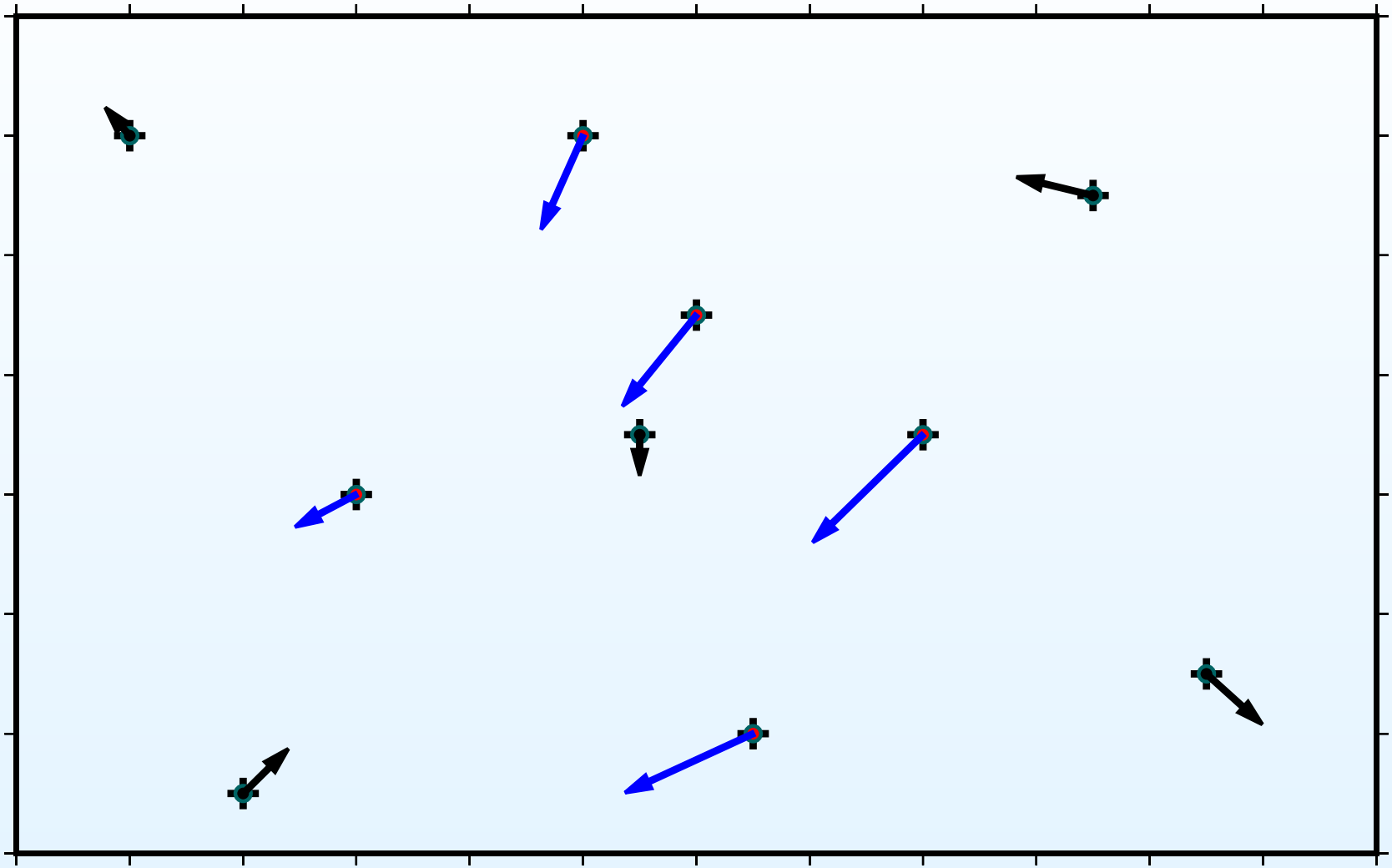
- S-transformation may be used to change (reduce) rank of covariance matrix

Uniform PPP results



All points obtain a coorection after transformation

Uniform PPP results



Transformed coordinates are congruent to uniform global system

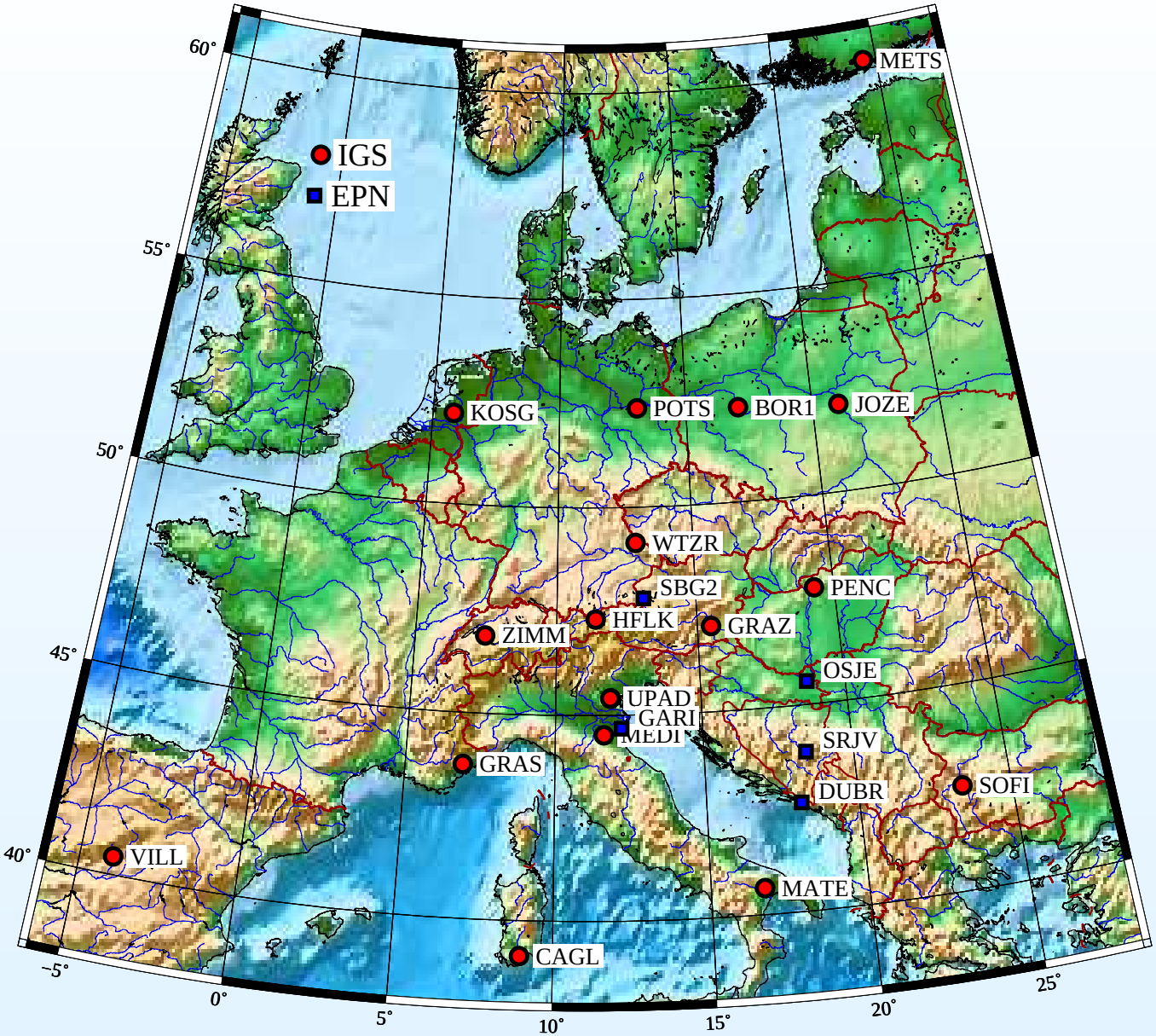
Case study

Case study

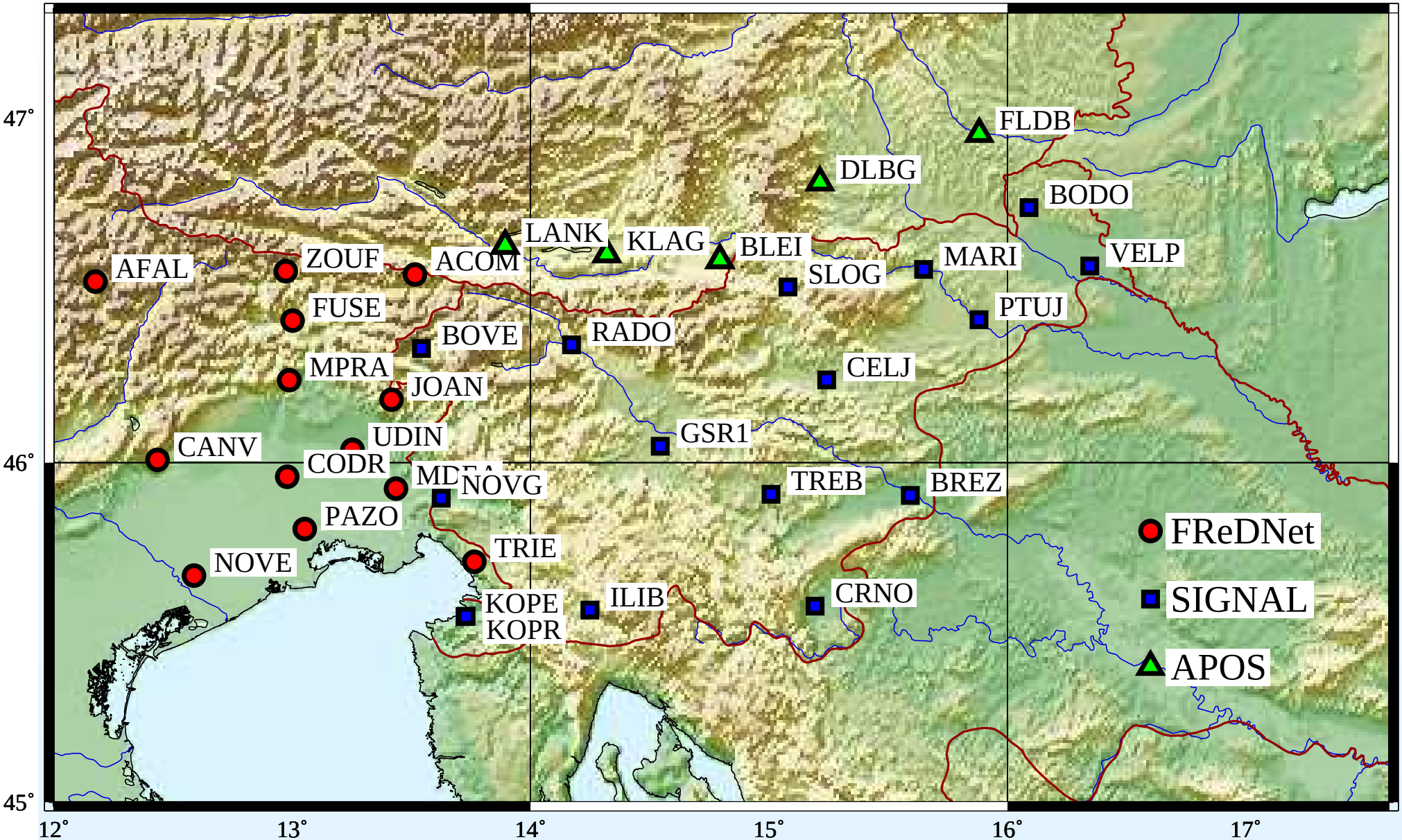
- GPS data from permanent station in/around Slovenia

Network	**#Stations**	**#Files**	**Time span**
SIGNAL	16	24 457	2002–2013
FReDNet	14	36 165	2002–2013
APOS	8	11 531	2003–2010
EPN	5	7 943	2002–2013
IGS	*18*	*71 696*	*2000–2013*
ALL	63	153 792	*2000–2013*

Permanent stations in Europe



Permanent stations in/around Slovenia



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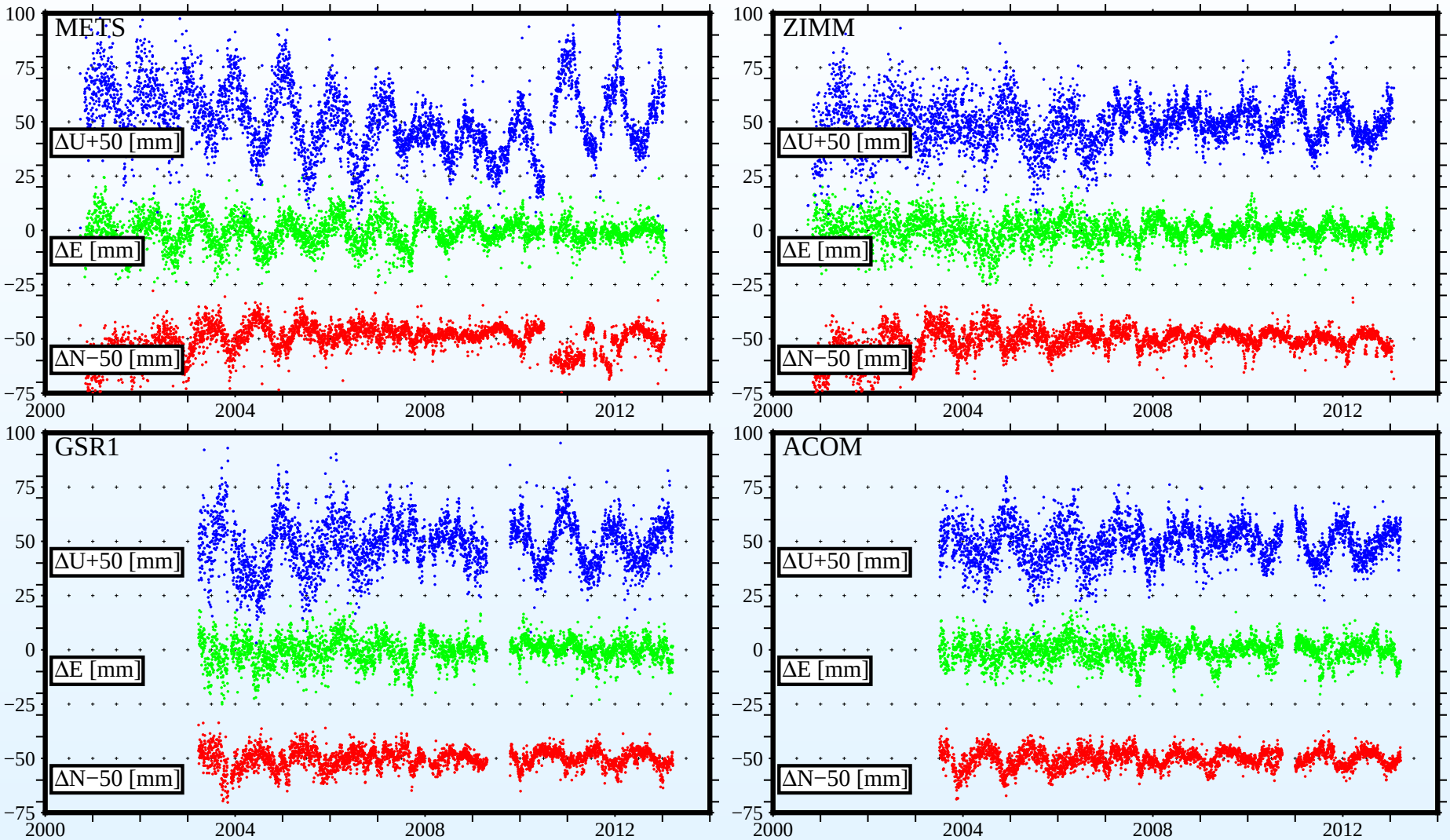
- Time series obtained for each station and each coordinate

Time series of daily coordinates



Repeatability of daily PPP coordinates

Network	$\bar{\sigma}_N$ [mm]	$\bar{\sigma}_E$ [mm]	$\bar{\sigma}_U$ [mm]
SIGNAL	4,4	5,9	10,2
FReDNet	4,2	5,7	9,5
APOS	4,4	5,5	10,8
EPN	4,5	5,6	9,9
IGS	6,2	6,8	11,8
All	*4,8*	*6,0*	*10,6*

- sub cm level of repeatability for N and E
- cm level of repeatability for U
- evident bias in coordinate time series

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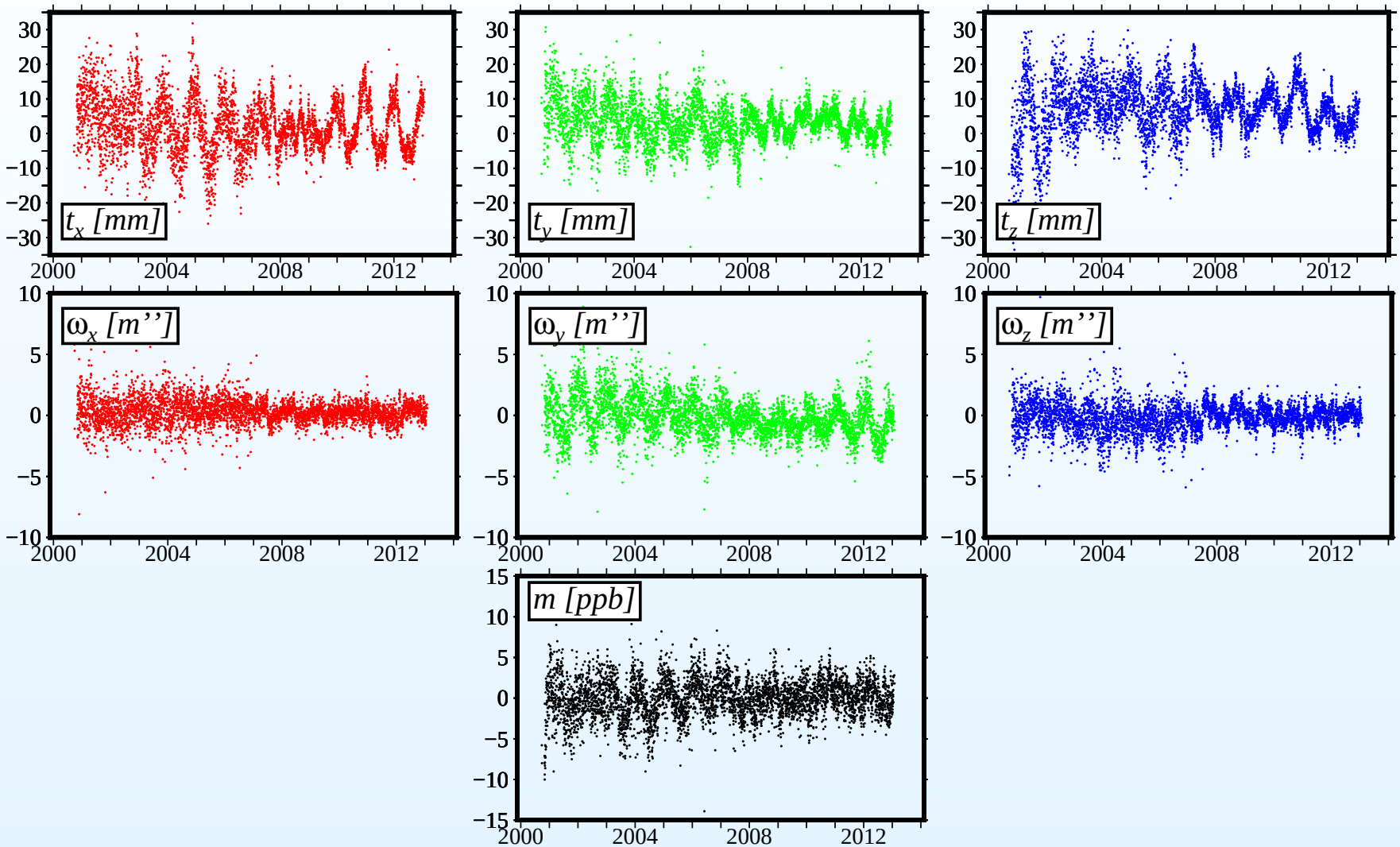
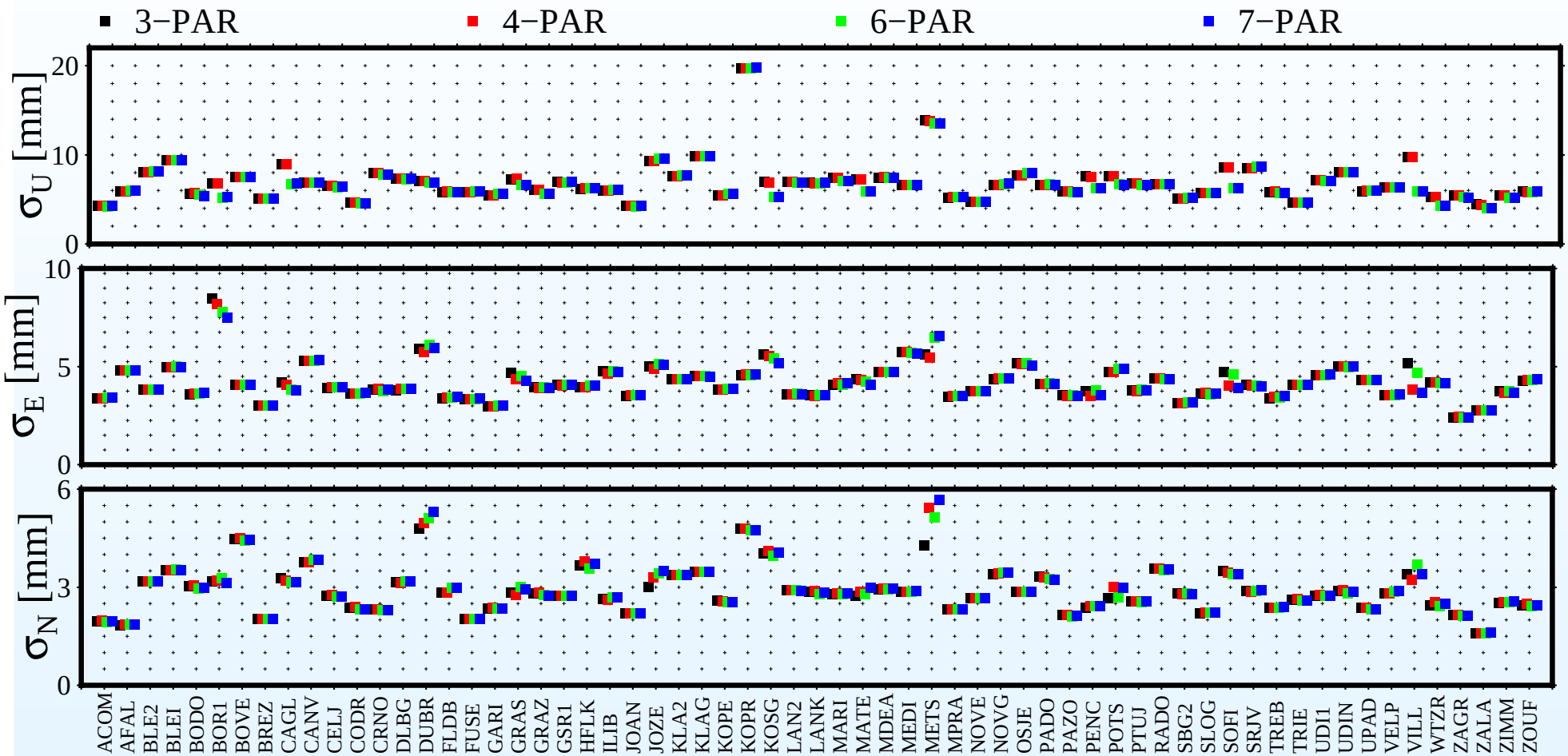
Transformacija	$\bar{\sigma}_N$ [mm]	$\bar{\sigma}_E$ [mm]	$\bar{\sigma}_U$ [mm]
3-parametric	3,1	5,1	7,8
4-parametric	3,1	4,9	7,8
6-parametric	3,1	5,0	6,4
7-parametric	3,1	4,8	6,4

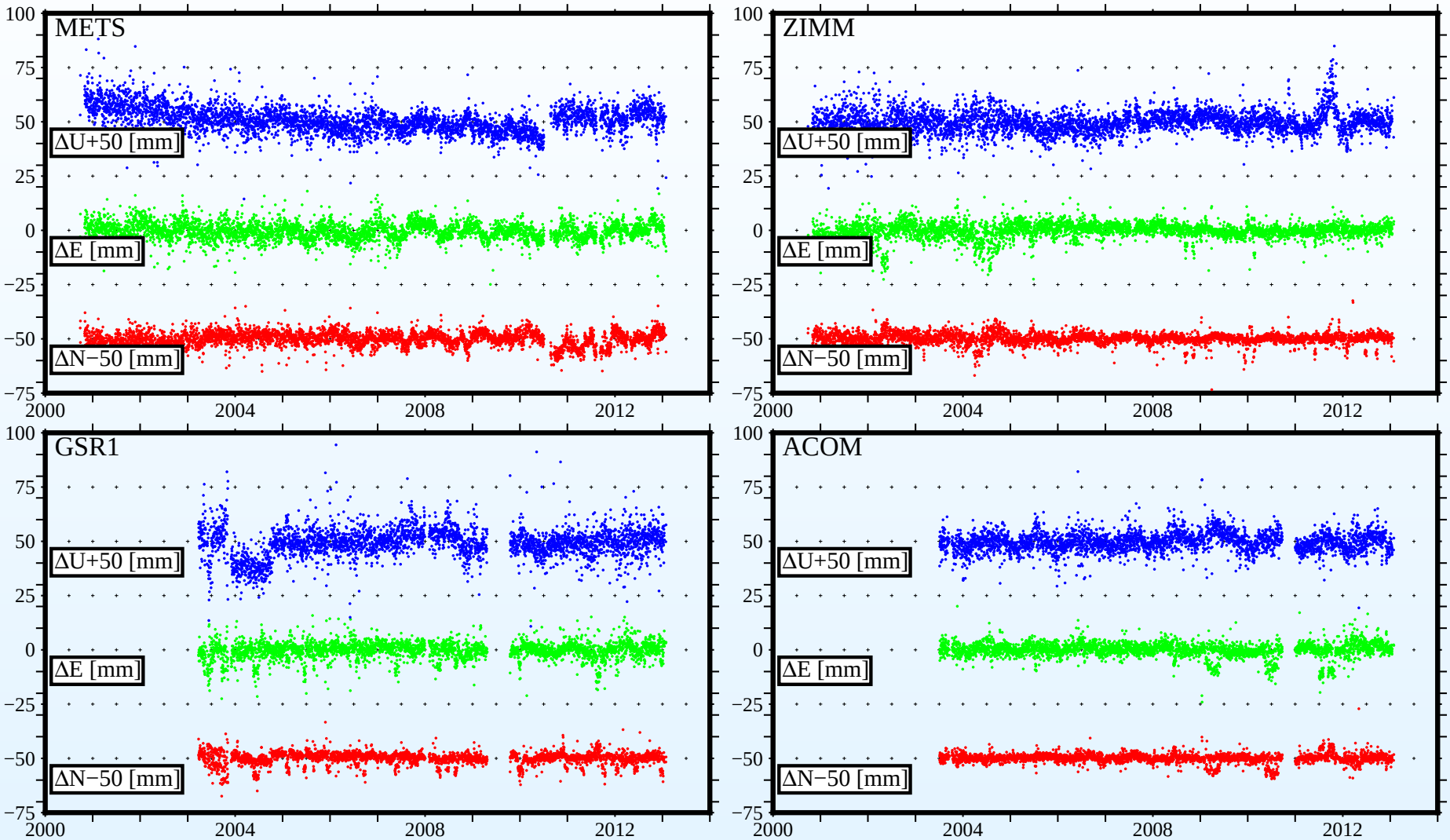
Estimated transformation parameters



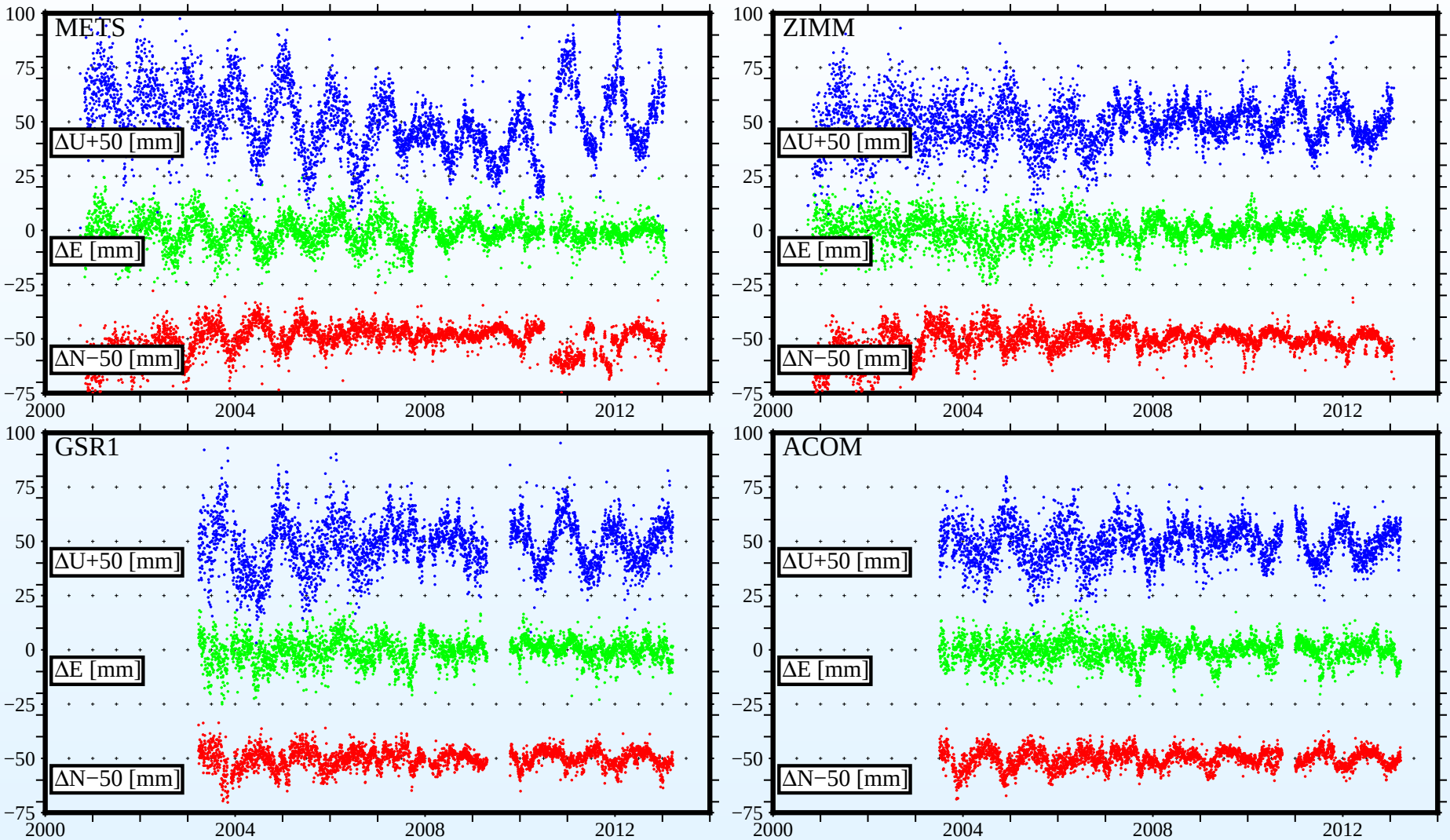
Impact of transf. parameters on coordinate repeatability



Transformed time series of daily coordinates



Original time series of daily coordinates



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	$\bar{\sigma}_N$ [mm]	$\bar{\sigma}_E$ [mm]	$\bar{\sigma}_U$ [mm]
PPP	**2,9**	**4,2**	**7,0**
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- Comparison to reference values

	σ_N [mm]	σ_E [mm]	σ_U [mm]
PPP–REF	**1,3**	**1,6**	**2,9**

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Precise point positioning:

- Analitical description of mathematical model (singularity, null space)
- Estimability of unknowns
- (Un)biasedness of estimated unknowns

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Uniformity of PPP results:

- Transformation represented with S-transformation
- Singularity of $\mathbf{\Sigma}_{\mathbf{X}^t}$
- Generalization of S-transformation to reduce rank

Summary

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Case study:

- Repeatability of PPP coordinates on cm level
- Evident bias in time series of PPP coordinates
- 3-parametric transformation sufficient
- Repeatability of transformed PPP coordinates on a few mm level
- Comparable to differential techniques (Bernese)
- Theoritical and practical equivalence of PPP results

. . . Thank you for your attention. . .